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The Heisenberg oscillator

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Abstract In this short note, we determine the spectrum of the Heisenberg oscillator which is the operator defined as $L+|x|^2+|y|^2$ on the Heisenberg group $H_1 = \mathbb{R}^2_{x,y} \times \mathbb{R}$ where L stands for the positive sublaplacian.

Keywords Nilpotent Lie groups · Harmonic oscillator · Representation of nilpotent Lie groups

1 Introduction

The quantum harmonic oscillator on the real line:

$$-\partial_x^2 + x^2,$$

is intimately linked with the three-dimensional real Heisenberg algebra $\mathfrak{h}_1$. Indeed on the one hand the operators of derivation ∂_x and of multiplication by ix generate the Heisenberg Lie algebra since their commutator $[\partial_x, ix] = i$ is central; on the other hand $-(-\partial_x^2 + x^2)$ is the sum of the square of these two operators.

This has the following well known consequences for the Heisenberg group $H_1 = \mathbb{R}^2 \times \mathbb{R}$ whose law is chosen here as:

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$$(x, y, t)(x', y', t') = \left(x + x', y + y', t + t' + \frac{xy' - x'y}{2}\right).$$

Let X, Y and T be the three elements of $\mathfrak{h}_1$ forming the canonical basis of $\mathfrak{h}_1$; it satisfies $[X, Y] = T$. We identify the elements of $\mathfrak{h}_1$ with left invariant vector fields on H_1 and we define the sublaplacian: $L = -(X^2 + Y^2)$. Let τ be the representation of H_1 on $L^2(\mathbb{R})$ such that

$$d\tau(X) = \partial_x, \ d\tau(Y) = ix \ \text{ and necessarily } \ d\tau(T) = i.$$

Then τ is the well known unitary irreducible Schrödinger representation of H_1 corresponding to the central character $t \mapsto e^{it}$. Furthermore

$$d\tau(L) = -\partial_x^2 + x^2.$$

The spectrum of the quantum harmonic is well known and this last equality allows to describe the spectrum of L.

In this short note, we reverse the line of approach described above to study the following unbounded operator on $L^2(H_1)$:

$$L + x^2 + y^2 = -(X^2 + Y^2) + x^2 + y^2;$$

we call this operator *the Heisenberg oscillator*. Our main result is the determination of its spectrum.

This study could very easily be generalised to the $(2n+1)$-dimensional Heisenberg group.

In fact we will study the operator $L+\lambda_2^2(x^2+y^2)$ for $\lambda_2 \neq 0$, even if by homogeneity it would suffice to study the case $\lambda_2 = 1$.

In the Heisenberg oscillator the central variable of H_1 appears only as derivatives in the expression of the vector fields

$$X = \partial_x - \frac{y}{2}\partial_t \quad \text{and} \quad Y = \partial_y + \frac{x}{2}\partial_t. \tag{1}$$

This motivates our choice to study the Heisenberg oscillator intertwined with the Fourier transform $\mathcal{F}_{\lambda_1}$ in the central variable of H_1:

$$\mathcal{F}_{\lambda_1} f(x, y) = \int_{\mathbb{R}} e^{-i\lambda_1 t} f(x, y, t) dt. \tag{2}$$

Hence the object at the centre of this paper is

$$\mathcal{F}_{\lambda_1}(L + \lambda_2^2(x^2 + y^2))\mathcal{F}_{\lambda_1}^{-1}, \tag{3}$$

where $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_2 \neq 0$.

The result of this note gives a complete description of the spectrum of the operator (3) which can also be viewed as a magnetic Schrödinger operator with quadratic potential. Some of the properties of the spectrum of that type of operators are already known by specialists of this domain (see for example [4]) and coincide with our explicit description in the particular case of the operator (3). In the future the result of this note will allow the study of a Mehler type formula for the operator given by (3), of the L^p-multipliers problem and of Strichartz estimates for the Heisenberg oscillator $L + (x^2 + y^2)$.

This paper is organised as follows. First we construct a six-dimensional nilpotent Lie group N and a representation ρ_λ of N such that the image of the canonical sublaplacian $\mathcal{L}$ of N through ρ_λ is given by (3). In the third section we study more systematically the representations of N via the orbit method and the diagonalisation of the image of $\mathcal{L}$. It allows us in the fourth section to go back to the study of the Heisenberg oscillator. In a last section, we obtain a Mehler type formula for the operator given by (3).

2 The nilpotent Lie group associated with the Heisenberg oscillator

2.1 The group N

We consider the unbounded operators on $L^2(H_1)$ given by the left-invariant vector fields X and Y (see (1)) and the multiplications by ix and iy. They generate a six-dimensional real Lie algebra

$$\mathfrak{n} := \mathbb{R}X_1 \oplus \mathbb{R}Y_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}Y_2 \oplus \mathbb{R}T_1 \oplus \mathbb{R}T_2,$$

whose canonical basis satisfies the commutator relations

$$[X_1, Y_1] = T_1,\ [X_1, X_2] = [Y_1, Y_2] = T_2,$$

with all the other commutators vanishing (beside the ones given by skew-symmetry). Hence $\mathfrak{n}$ is a well defined two-step nilpotent Lie algebra. It is stratified [3] since we can decompose:

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z},$$

where the subspace

$$\mathfrak{v} := \mathbb{R}X_1 \oplus \mathbb{R}Y_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}Y_2,$$

generates the Lie algebra $\mathfrak{n}$ and the subspace

$$\mathfrak{z} := \mathbb{R}T_1 \oplus \mathbb{R}T_2,$$

is the centre of $\mathfrak{n}$.

The connected simply connected nilpotent Lie group associated with $\mathfrak{n}$ is N identified with $\mathfrak{v} \times \mathfrak{z} \sim \mathbb{R}^6$ using exponential coordinates. Hence N is endowed with the group law

$$(v, z)(v', z') = (v + v', z'')$$

where, for $v = (x_1, y_1, x_2, y_2)$, $v = (x_1', y_1', x_2', y_2')$, $z = (z_1, z_2)$ and $z' = (z_1', z_2')$, we have:

$$z'' = \left(z_1 + z_1' + \frac{x_1 y_1' - x_1' y_1}{2}, z_2 + z_2' + \frac{x_1 x_2' - x_2 x_1'}{2} + \frac{y_1 y_2' - y_2 y_1'}{2} \right).$$

We identify the elements of $\mathfrak{n}$ with left invariant vector fields on N. We denote by

$$\mathcal{L} := -(X_1^2 + Y_1^2 + X_2^2 + Y_2^2), \tag{4}$$

the canonical sublaplacian of N.

2.2 The representation ρ_λ

Let $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_2 \neq 0$. We consider the representation $d\rho_\lambda$ of the Lie algebra $\mathfrak{n}$ over $L^2(\mathbb{R}^2)$ defined by:

$$\begin{cases} d\rho_\lambda(X_1) = \mathcal{F}_{\lambda_1} X \mathcal{F}_{\lambda_1}^{-1} = \partial_x - i\frac{y}{2}\lambda_1 \\ d\rho_\lambda(Y_1) = \mathcal{F}_{\lambda_1} Y \mathcal{F}_{\lambda_1}^{-1} = \partial_y + i\frac{x}{2}\lambda_1 \\ d\rho_\lambda(X_2) = i\lambda_2 x \quad d\rho_\lambda(Y_2) = i\lambda_2 y \\ d\rho_\lambda(T_1) = i\lambda_1 \quad d\rho_\lambda(T_2) = i\lambda_2 \end{cases}. \tag{5}$$

Throughout this paper, $L^2(\mathbb{R}^2)$ is endowed with its natural Hilbert space structure whose Hermitian product is given by:

$$(f, g)_{L^2(\mathbb{R}^2)} = \int f(x, y)\bar{g}(x, y)dxdy.$$

It is not difficult to compute that $d\rho_\lambda$ is the infinitesimal representation of the unitary representation ρ_λ of N on $L^2(\mathbb{R}^2)$ given by:

$$\rho_\lambda(v, z) f(x, y) = e^{i\lambda_1(z_1 + \frac{xy_1 - x_1 y}{2}) + i\lambda_2(z_2 + xx_2 + yy_2 + \frac{x_1 x_2}{2} + \frac{y_1 y_2}{2})} f(x + x_1, y + y_1),$$

where $f \in L^2(\mathbb{R}^2)$, $(x, y) \in \mathbb{R}^2$, $(v, z) \in N$ with $v = (x_1, y_1, x_2, y_2)$ and $z = (z_1, z_2)$.

By (5) the image of the canonical sublaplacian $\mathcal{L}$ of N (see (4)) through ρ_λ is:

$$d\rho_\lambda(\mathcal{L}) = \mathcal{F}_{\lambda_1}(L + \lambda_2^2(x^2 + y^2))\mathcal{F}_{\lambda_1}^{-1}. \tag{6}$$

In the next section, we will show that ρ_λ is equivalent to an irreducible unitary representation π_λ and we will diagonalise $\pi_\lambda(\mathcal{L})$.

3 The representations of N

In this section, after describing all the unitary irreducible representations of N using the orbit method [1], we obtain a diagonalisation of $\rho_\lambda(\mathcal{L})$.

3.1 All the representations of N

We need to describe the orbits of N acting on the dual $\mathfrak{n}^*$ of $\mathfrak{n}$ by the dual of the adjoint action. Each element of $\mathfrak{n}^*$ will be written as $\ell = (\omega, \lambda)$ where ω and λ are linear forms on $\mathfrak{v}$ and $\mathfrak{z}$ respectively, identified with a vector of $\mathfrak{v}$ and $\mathfrak{z}$ by the canonical scalar products of these two spaces. It is not difficult to determine representatives of the co-adjoint orbits:

Lemma 3.1 *Each co-adjoint orbit of N admits exactly one representative of the form* $\ell = (\omega, \lambda)$ *with* $\lambda = (\lambda_1, \lambda_2)$ *satisfying*

(i) $\lambda_2 \neq 0$ *and* $\omega = 0$
(ii) $\lambda_2 = 0, \lambda_1 \neq 0, \omega \in \mathbb{R}X_2 \oplus \mathbb{R}Y_2$
(iii) $\lambda_1 = \lambda_2 = 0$ *and any* ω.

Sketch of the proof For each $z \in \mathfrak{z}$, let j_z be the endomorphism of $\mathfrak{v}$ given by:

$$\langle j_z(v), v' \rangle_{\mathfrak{v}} = \langle z, [v, v'] \rangle_{\mathfrak{z}}, \quad v, v' \in \mathfrak{v},$$

where $\langle , . \rangle_{\mathfrak{v}}$ and $\langle , . \rangle_{\mathfrak{z}}$ denote the canonical scalar products on $\mathfrak{v}$ and $\mathfrak{z}$ respectively. In the canonical basis $\{X_1, Y_1, X_2, Y_2\}$ of $\mathfrak{v}$, the endomorphism j_z is represented by:

$$\begin{pmatrix} 0 & z_1 & z_2 & 0 \\ -z_1 & 0 & 0 & z_2 \\ -z_2 & 0 & 0 & 0 \\ 0 & -z_2 & 0 & 0 \end{pmatrix} \quad \text{whose determinant is } z_2^4.$$

So the the range of j_z is $\mathfrak{v}$ if $z_2 \neq 0$, $\mathbb{R}X_1 \oplus \mathbb{R}Y_1$ if $z_2 = 0$ but $z_1 \neq 0$.

As the nilpotent Lie group N is of step two, we compute easily for $\ell = (\omega, \lambda)$ and $n = (v_o, z_o) \in N$:

$$\ell \circ \mathrm{Ad}(n^{-1}) = (\omega + j_\lambda(v_o), \lambda),$$

and the previous paragraph completes the proof. □

It is a routine exercise to compute a representation associated with a linear form and we just give here the end result for the linear forms ℓ described in Lemma 3.1.

Let $\lambda=(\lambda_1,\lambda_2)$ with $\lambda_2\neq 0$ as in (i) of Lemma 3.1. The representation π_λ of N over $L^2(\mathbb{R}^2)$ given by:

$$\begin{aligned}\pi_\lambda(v,t)h(u_1,u_2) &= e^{i\lambda_1(t_1+u_1y_1+\frac{x_1y_1}{2})+i\lambda_2(t_2+u_1x_2-u_2y_1+\frac{x_1x_2}{2}-\frac{y_1y_2}{2})}\\ &\quad\times h(x_1+u_1,y_2+u_2),\end{aligned}$$

is the irreducible unitary representation associated with the linear form given by λ (for the polarisation $\mathbb{R}Y_1\oplus\mathbb{R}X_2\oplus\mathbb{R}T_1\oplus\mathbb{R}T_2$).

Let $\lambda_2=0$, $\lambda_1\neq 0$, $\omega\in\mathbb{R}X_2\oplus\mathbb{R}Y_2$ as in (ii) of Lemma 3.1. The representation $\pi_{\lambda_1,\omega}$ of N over $L^2(\mathbb{R})$ given by:

$$\pi_{\lambda_1,\omega}(v,t)h(u)=\exp i\lambda_1(t_1+u_1y_1+\frac{1}{2}x_1y_1)\ \exp i\langle\omega,v\rangle\ h(x_1+u),$$

is the irreducible unitary representation associated with the linear form given by (ω,λ).

Let $\lambda_2=\lambda_1=0$ and $\omega\in\mathfrak{v}$ as in (iii) of Lemma 3.1. The character

$$e^{i\langle\omega,\cdot\rangle}:(v,t)\longmapsto\exp i\langle\omega,v\rangle,$$

gives the one-dimensional unitary representation associated with the linear form given by ω.

By Kirillov's methods, the representations $\pi_\lambda,\pi_{\lambda_1,w}$ and $e^{i\langle\omega,\cdot\rangle}$ exhaust all the irreducible unitary representations of N, up to unitary equivalence.

3.2 The representations π_λ and ρ_λ

Let us focus on the representations π_λ with $\lambda=(\lambda_1,\lambda_2)$, $\lambda_2\neq 0$. Its infinitesimal representation is given by:

$$\begin{cases} d\pi_\lambda(X_1)=\partial_{u_1} & d\pi_\lambda(Y_1)=i\lambda_1u_1-i\lambda_2u_2\\ d\pi_\lambda(X_2)=i\lambda_2u_1 & d\pi_\lambda(Y_2)=\partial_{u_2}\\ d\pi_\lambda(T_1)=i\lambda_1 & d\pi_\lambda(T_2)=i\lambda_2\end{cases}. \tag{7}$$

We can now go back to the study of the representation ρ_λ. Its restriction to the centre gives the character $z\mapsto e^{i\lambda(z)}$; so by Kirillov's method, we know that ρ_λ is equivalent to one or several copies of π_λ, depending whether ρ_λ is irreducible. In fact it is not difficult to find a concrete expression for the intertwiner between ρ_λ and π_λ (see the proposition just below) and this shows in particular that ρ_λ is irreducible.

Proposition 3.2 *For each $\lambda=(\lambda_1,\lambda_2)$, $\lambda_2\neq 0$, the representations ρ_λ and π_λ are unitarily equivalent. More precisely, let $T_\lambda=T:L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ be the unitary operator given by:*

$$Th(x,y)=\sqrt{\frac{|\lambda_2|}{2\pi}}\,e^{i\frac{\lambda_1}{2}xy}\int_{\mathbb{R}}e^{-i\lambda_2yz}h(x,z)dz.$$

Then

$$T\pi_\lambda = \rho_\lambda T.$$

Proof The operator T can be written as $T = T_1 T_2$ where $T_1, T_2 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ are the unitary operators given by:

$$T_1 f(x, y) = e^{i\frac{\lambda_1}{2}xy} f(x, y)$$
$$T_2 f(x, v) = \sqrt{\frac{|\lambda_2|}{2\pi}} \int_{\mathbb{R}} e^{-i\lambda_2 vy} f(x, y) dy.$$

The computations of the infinitesimal action on the canonical basis through $\rho_\lambda^{(1)} = T_1^{-1} \circ \rho_\lambda \circ T_1$ and then $\rho_\lambda^{(2)} = T_2^{-1} \circ \rho_\lambda^{(1)} \circ T_2$ yield the result. □

3.3 Diagonalisation of $d\pi_\lambda(\mathcal{L})$

By (7) the image of the canonical sublaplacian through π_λ is the operator:

$$d\pi_\lambda(\mathcal{L}) = -\partial_{u_1}^2 + (\lambda_1 u_1 - \lambda_2 u_2)^2 + (\lambda_2 u_1)^2 - \partial_{u_2}^2,$$

for which we determine a diagonalisation basis.

We need to study the homogeneous polynomial of degree two:

$$(\lambda_1 u_1 - \lambda_2 u_2)^2 + (\lambda_2 u_1)^2 = u^t M_\lambda u, \tag{8}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad M_\lambda = \begin{pmatrix} \lambda_1^2 + \lambda_2^2 & -\lambda_1\lambda_2 \\ -\lambda_1\lambda_2 & \lambda_2^2 \end{pmatrix},$$

and this boils down to diagonalising the matrix M_λ. We obtain:

$$k_\lambda^{-1} M_\lambda k_\lambda = \begin{pmatrix} \mu_{+,\lambda} & 0 \\ 0 & \mu_{-,\lambda} \end{pmatrix}$$

where

$$\mu_{\epsilon,\lambda} = \frac{1}{2}\left(\lambda_1^2 + 2\lambda_2^2 + \epsilon|\lambda_1|\sqrt{\lambda_1^2 + 4\lambda_2^2}\right) > 0, \quad \epsilon = \pm, \tag{9}$$

and k_λ is the orthogonal 2×2-matrix:

$$k_\lambda = \begin{pmatrix} \frac{\lambda_1\lambda_2}{\sqrt{(\lambda_1\lambda_2)^2 + \left(\frac{\lambda_1^2 - |\lambda_1|\sqrt{\lambda_1^2+4\lambda_2^2}}{2}\right)^2}} & \frac{\lambda_1\lambda_2}{\sqrt{(\lambda_1\lambda_2)^2 + \left(\frac{\lambda_1^2 + |\lambda_1|\sqrt{\lambda_1^2+4\lambda_2^2}}{2}\right)^2}} \\ \frac{\lambda_1^2 - |\lambda_1|\sqrt{\lambda_1^2+4\lambda_2^2}}{2\sqrt{(\lambda_1\lambda_2)^2 + \left(\frac{\lambda_1^2 - |\lambda_1|\sqrt{\lambda_1^2+4\lambda_2^2}}{2}\right)^2}} & \frac{\lambda_1^2 + |\lambda_1|\sqrt{\lambda_1^2+4\lambda_2^2}}{2\sqrt{(\lambda_1\lambda_2)^2 + \left(\frac{\lambda_1^2 + |\lambda_1|\sqrt{\lambda_1^2+4\lambda_2^2}}{2}\right)^2}} \end{pmatrix}. \tag{10}$$

The change of variable

$$u' = k_\lambda u, \quad u' = \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{11}$$

transforms the homogeneous polynomial (8) into $\mu_{+,\lambda} {u'_1}^2 + \mu_{-,\lambda} {u'_2}^2$ and leaves the 2-dimensional laplacian invariant, that is, $-(\partial_{u_1}^2 + \partial_{u_2}^2) = -(\partial_{u'_1}^2 + \partial_{u'_2}^2)$; the operator $\pi_\lambda(\mathcal{L})$ becomes:

$$\pi_\lambda(\mathcal{L}) = -\partial_{u'_1}^2 - \partial_{u'_2}^2 + \mu_{+,\lambda} {u'_1}^2 + \mu_{-,\lambda} {u'_2}^2, \quad u' = k_\lambda u. \tag{12}$$

Recall that the Hermite functions h_m, $m \in \mathbb{N}$, defined by:

$$h_m(x) = e^{-\frac{x^2}{2}} H_m(x) \quad \text{where} \quad H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m}(e^{-x^2}),$$

form an orthonormal basis of $L^2(\mathbb{R})$ which diagonalises the quantum harmonic oscillator:

$$-h''_m(x) + x^2 h_m = (2m+1) h_m.$$

Using the notation above, we obtain:

Proposition 3.3 *The operator $\pi_\lambda(\mathcal{L})$ admits the following orthonormal basis of eigenfunctions:*

$$h_{\lambda,m}(u) := |\lambda_2|^{-1/2} h_{m_+}(\mu_{+,\lambda}^{1/4} u'_1)\, h_{m_-}(\mu_{-,\lambda}^{1/4} u'_2)$$

where $m = (m_+, m_-) \in \mathbb{N}^2$ and $u' = k_\lambda u$. The eigenvalue associated with $h_{\lambda,m}$ is

$$\nu_{\lambda,m} := \mu_{+,\lambda}^{1/2}(2m_+ + 1) + \mu_{-,\lambda}^{1/2}(2m_- + 1).$$

Consequently, by Proposition 3.2, we obtain:

Corollary 3.4 *The operator given by* (6)*, that is,*

$$d\rho_\lambda(\mathcal{L}) = \mathcal{F}_{\lambda_1}(L + \lambda_2^2(x^2 + y^2))\mathcal{F}_{\lambda_1}^{-1},$$

admits $\{Th_{\lambda,m},\ m \in \mathbb{N}^2\}$ *as orthonormal basis of eigenfunctions and the eigenvalue associated with* $Th_{\lambda,m}$ *is* $\nu_{\lambda,m}$.

4 Spectrum of $L - \lambda_2^2(x^2 + y^2)$

For any $f \in L^2(H_1)$, $\lambda = (\lambda_1, \lambda_2)$, $\lambda_2 \neq 0$, and $m \in \mathbb{N}^2$, we define:

$$c_{\lambda,m}(f) := (\mathcal{F}_{\lambda_1} f, Th_{\lambda,m})_{L^2(\mathbb{R}^2)}, \tag{13}$$

where $\mathcal{F}_{\lambda_1}$ is the Fourier transform (2) in the central variable and $Th_{\lambda,m}$ the orthonormal basis of $L^2(\mathbb{R}^2)$ given in Corollary 3.4.

Lemma 4.1 *We have for any* $f \in L^2(H_1)$ *such that* $(L + \lambda_2^2(x^2 + y^2))f \in L^2(H_1)$*:*

$$c_{\lambda,m}((L + \lambda_2^2(x^2 + y^2))f) = \nu_{l,m} c_{\lambda,m}(f).$$

Proof Recall

$$\mathcal{F}_{\lambda_1}((L + \lambda_2^2(x^2 + y^2))f) = d\rho_\lambda(\mathcal{L})\mathcal{F}_{\lambda_1} f.$$

As $d\rho(\mathcal{L})$ is self-adjoint, we have:

$$\begin{aligned}
c_{\lambda,m}((L + \lambda_2^2(x^2 + y^2))f) &= (d\rho_\lambda(\mathcal{L})\mathcal{F}_{\lambda_1} f, Th_{\lambda,m})_{L^2(\mathbb{R}^2)} \\
&= (\mathcal{F}_{\lambda_1} f, d\rho_\lambda(\mathcal{L})Th_{\lambda,m})_{L^2(\mathbb{R}^2)} = \bar{\nu}_{l,m}(\mathcal{F}_{\lambda_1} f, Th_{\lambda,m})_{L^2(\mathbb{R}^2)} \\
&= \nu_{l,m} c_{\lambda,m}(f),
\end{aligned}$$

by Corollary 3.4. □

Now we fix $\lambda_2 \in \mathbb{R}\backslash\{0\}$. For any Borelian set B of $\mathbb{R}$, let $E(B)$ be the operator defined on $L^2(H_1)$ by

$$E(B)f = \mathcal{F}_{\lambda_1}^{-1}\left[\sum_{m\in\mathbb{N}^2} 1_{\nu_{\lambda,m}\in B} c_{\lambda,m}(f) Th_{\lambda,m}\right],$$

where $c_{\lambda,m}(f)$ is defined by (13). With Lemma 4.1, it is a routine exercise to check that $B \mapsto E(B)$ is the spectral resolution of $L + \lambda_2^2(x^2 + y^2)$. The spectrum is:

$$\{\nu_{(\lambda_1,\lambda_2),m}, \lambda_1 \in \mathbb{R},\ m \in \mathbb{N}^2\} = [\nu_{(0,\lambda_2),0}, +\infty),$$

where

$$\nu_{(0,\lambda_2),0} = \mu_{+,(0,\lambda_2)}^{1/2} + \mu_{-,(0,\lambda_2)}^{1/2} = 2|\lambda_2|.$$

5 Application: Mehler type formulae

The Mehler formula [2, Theorem.12.63] states that the integral kernel of the operator $\exp(-t(-\partial_x^2 + x^2 - 1))$ is:

$$Q_t(x, y) = \pi^{-\frac{1}{2}}(1 - e^{-4t})^{-\frac{1}{2}} \exp(-F_t(x, y)),$$

where

$$F_t(x, y) = (1 - e^{-4t})^{-1} \left(\frac{1}{2}(1 + e^{-4t})(x^2 + y^2) - 2e^{-2t} xy \right).$$

Hence for any $\mu > 0$, the integral kernel of $\exp(-t(-\partial_x^2 + \mu x^2))$ is:

$$K_{t,\mu}(x, y) = \sqrt{\mu} e^{-t\mu} Q_{t\mu}(\sqrt{\mu}x, \sqrt{\mu}y).$$

We conclude this note with the following Mehler type formulae for the operators $d\pi_\lambda(\mathcal{L})$ and $d\rho_\lambda(\mathcal{L}) = \mathcal{F}_{\lambda_1}(L + \lambda_2^2(x^2 + y^2))\mathcal{F}_{\lambda_1}^{-1}$ (given by (6)):

Proposition 5.1 *The integral kernel of the operator* $\exp(-td\pi_\lambda(\mathcal{L}))$ *is:*

$$\kappa_{t,\lambda}((u_1, u_2), (v_1, v_2)) = K_{t,\mu_{+,\lambda}}(u_1, v_1) K_{t,\mu_{-,\lambda}}(u_2, v_2).$$

The integral kernel of the operator $\exp(-td\rho_\lambda(\mathcal{L}))$ *is:*

$$\begin{aligned} &Q_{t,\lambda}((x_o, y_o), (x, y)) \\ &\quad = \frac{|\lambda_2|}{2\pi} e^{i\frac{\lambda_1}{2}(x_o y_o - xy)} \int_{\mathbb{R}^2} e^{i\lambda_2(y_2 y - y_o y_1)} \kappa_{t,\lambda}((x_o, y_1), (x, y_2)) dy_1 dy_2 \end{aligned}$$

Proof The first formula is easily obtained from (12).

For the second formula, we see that, by Proposition 3.2, we have:

$$\exp(-td\rho_\lambda(\mathcal{L})) = T \exp(-td\pi_\lambda(\mathcal{L}))T^{-1},$$

the operators T and T^{-1} having integral kernels:

$$\begin{aligned} C_T((x, y), (x', y')) &= \sqrt{\frac{|\lambda_2|}{2\pi}} e^{i\frac{\lambda_1}{2}xy} e^{-i\lambda_2 yy'} \delta_{x'=x}, \\ C_{T^{-1}}((x, y), (x', y')) &= \sqrt{\frac{|\lambda_2|}{2\pi}} e^{-i\frac{\lambda_1}{2}xy'} e^{i\lambda_2 yy'} \delta_{x'=x}. \end{aligned}$$

So the operator $\exp(-td\rho_\lambda(\mathcal{L}))$ has integral kernel:

$$
\begin{aligned}
&Q_{t,\lambda}((x_o,y_o),(x,y))\\
&= \int C_T((x_o,y_o),(x_1,y_1))\kappa_{t,\lambda}((x_1,y_1),(x_2,y_2))C_{T^{-1}}((x_2,y_2),(x,y))\\
&\qquad\qquad dx_1dy_1dx_2dy_2\\
&= \frac{|\lambda_2|}{2\pi}\int\limits_{\mathbb{R}^2} e^{i\frac{\lambda_1}{2}x_oy_o}e^{-i\lambda_2y_oy_1}\kappa_{t,\lambda}((x_o,y_1),(x,y_2))e^{-i\frac{\lambda_1}{2}xy}e^{i\lambda_2y_2y}dy_1dy_2
\end{aligned}
$$

□

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